(c) And so on for all other values of p, q, and m.

Mr. Gruber sent in three different solutions, and Mr. D. B. Northrup, of Mandana, N. Y., sent in the results for tracts in the form of (1) circle, (2) square, (3) rectangle sides as 2:1, (4) a triangle ratios of side 6:6:7, and (5) an ellipse having a major-axis double the minor-axis. Other very simple solutions of the problem are possible when the tract is in the shape of a square, but the solutions above are quite sufficient for all purposes. ED. F.

136. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

What is the size of the smallest cubical box, inside dimension, that will contain four balls each ten inches in diameter?

Solution by H. N. DAVIS, (Brown University), 159 Brown Street, Providence, R. I.

Let the base of the box be of such a size that two of the balls when placed with their centers along a diagonal will be tangent to each other and to the sides of the box. Let their centers be C and C', the diagonal AD, and the center O.

Draw BC perpendicular to AE. Then $AB=BC=r=5$.

$\therefore AC=5\sqrt{2}$.

$AO=5\sqrt{2}+CO=5\sqrt{2}+r=5\sqrt{2}+5$.

$AD=10(1+\sqrt{2})=\sqrt{2}AE$.

$AE=5\sqrt{2}+10=17.07106$ inches (nearly).

If the second layer of two balls be placed along the other diagonal (at $M+M'$) the position of M with reference to C and the side AF considered as a base will be exactly the same as that of C and C' with reference to AD. The figure may then be taken as an elevation and the height of the box will be exactly equal to its base-edge, and the box will be a cube. Q. E. D.

Good solutions were received from *PROF. J. M. STRASBURG*, Chicago, Ill., *D. B. NORTHRUP*, *G. B. M. ZERR*, and *M. A. GRUBER*.

ALGEBRA.

112. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In *Hall and Knight's Higher Algebra* I find the following:

If $a+b+c=0$, then

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2};\qquad \frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2};$$

and if $a+b+c+d=0$, then

$$\frac{a^5+b^5+c^5+d^5}{5}=\frac{a^3+b^3+c^3+d^3}{3}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

QUERY. Is there a general law governing such expressions? Investigate.

I. Solution by HARRY S. VANDIVER, Bala, Montgomery County, Pa.

Consider the following general equation with its second term missing, thus

$$f(x)=x^n+m_2x^{n-2}-m_3x^{n-3}\ldots\ldots m_{n-1}x+(-1)^n m_n=0\ldots\ldots(1).$$

Suppose the roots of this equation to be a_1, a_2, a_3, $\ldots\ldots a_n$.

Then $f'(x)=\dfrac{f(x)}{x-a_1}+\dfrac{f(x)}{x-a_2}+\dfrac{f(x)}{x-a^3}\ldots\ldots\dfrac{f(x)}{x-a_n}\ldots\ldots(2).$

[*Hall and Knight's Higher Algebra*, page 468.]

Divide (1) by $x-a_1$,

$$\frac{f(x)}{x-a_1}=x^{n-1}+a_1x^{n-2}+(m_2+a_1^2)x^{n-3}-(m_3-a_1m_2-a_1^3)x^{n-4}\ldots\ldots;$$

the general term being

$$(-1)^r[m_r-a_1m_{r-1}\ldots\ldots(-1)^r a_1^{r-2}m_2+(-1)^r a_1^r]x^{n-(r+1)};$$

and similar expressions for

$$\frac{f(x)}{x-a_2},\ \frac{f(x)}{x-a_3}\ldots\ldots\frac{f(x)}{x-a_n}.$$

Finding the value of the right-hand member of (2) by addition of the expressions just obtained, and adopting the notation

$$S_h=\frac{a_1^h+a_2^h\ldots\ldots a_n^h}{h},$$

we get, $f'(x)=nx^{n-1}+m_2(n-2)x^{n-3}-m_3(n-3)x^{n-4}\ldots\ldots m_{n-2}x+(-1)^{n-1}m_{n-1}$

$$=nx^{n-1}+(nm_2+2S_2)x^{n-3}-(nm_3-3S_3)x^{n-4}+(nm_4+2m_2S_2+4S_4)x^{n-5}\ldots\ldots;$$

the general term being

$$(-1)^r[nm_r+2m_{r-2}S_2-3m_{r-3}S_3\ldots\ldots(-1)^r(r-2)m_2S_{r-2}+(-1)^r rS_r].$$

Whence, by equating coefficients, and solving for m_2, m_3 $\ldots m_{n-1}$) we obtain,

$$m_2=S_2\ldots\ldots(5);\ m_3=S_3\ldots\ldots(6);\ m_4=\frac{-(2m_2S_2+4S_4)}{4}\ldots\ldots(7);$$

$$m_5=\frac{-(2m_3S_2-3m_2S_3-5S_5)}{5}\ldots\ldots(8);$$

and in general,

$$m_r=\frac{-[2m_{r-2}S_2-3m_{r-3}S_3\ldots\ldots(-1)^r(r-2)m_2S_{r-2}+(-1)^r rS_r]}{r}\ldots\ldots(9).$$

Multiplying (1) by x^{k-n}, we have

$$x^k+m_2x^{k-2}-m_3x^{k-3}\ldots.(-1)^r m_r x^{k-r}\ldots.(-1)^n m_n x^{k-n}=0.$$

Substituting a_1, a_2, $a_3\ldots.a_n$ each separately for x, and adding the identities thus obtained, we have

$$kS_k+(k-2)m_2S_{k-2}-(k-3)m_3S_{k-3}\ldots.(-1)^r(k-r)m_rS_{k-r}\ldots.(-1)^n\ldots.(10).$$

$(k-n)m_nS_{k-n}=0$.

By substituting the value of m_2 in (7) we obtain $S_2^2/2-S_4=m_4$.

In like manner, by successive substitutions, m_5, $m_6\ldots.m_{n-1}$ can be obtained in terms of S_2, $S_3\ldots.S_{n-1}$.

Hence by substitution in (10) we may obtain a relation satisfying the conditions of the problem, for any positive integral value of n.

When $n=3$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3};$$

putting $k=5$, then $S_5=S_3\times S^2$; when $k=7$, then $S_7=S_5\times S_2$.

. . . . .

When $n=4$,

$$kS_k=(k-2)S_2S_{k-2}+(k-3)S_3S_{k-3}+(k-4)(S_4-S_2^2/2)S_{k-4};$$

putting $k=5$, $S_5=S_3\times S_2$.

. . . . .

When $n=5$, we have

$$kS_k-(k-2)S_2S_{k-2}-(k-3)S_3S_{k-3}+(k-4)$$
$$(S_2^2/2-S_4)S_{k-4}-(S_5-S_2S_3)(k-5)S_{k-5}=0.$$

II. Solution by ROBERT B. HAYWARD, Ashcombe, Shanklin, Isle of Wight.

S_r denotes $a^r+b^r+c^r$.

Then $S_1=a+b+c=0$.

$S_2=a^2+b^2+c^2=-2(bc+ca+ab)$.

$S_3=a^3+b^3+c^3=3abc$.

Hence a, b, c are the roots of the equation.

$$x^3-\frac{S_2}{2}.x-\frac{S_3}{3}=0.$$

$$\therefore\ x^n-\frac{S_2}{2}.x^{n-2}-\frac{S_3}{3}x^{n-3}=0.$$

Substitute a, b, c successively for x, and add, then

$$S_n - \frac{S_2}{2}.S_{n-2} - \frac{S_3}{3}.S_{n-3} = 0.$$

Whence S_n is determinable in powers of $\frac{S_2}{2}$ and $\frac{S_3}{3}$.

Hence $S_5 = \frac{S_2}{2}.S_3 + \frac{S_3}{3}.S_2 = \frac{5}{6}S_3.S_2$, or $\frac{S_5}{5} = \frac{S_3}{3}.\frac{S_2}{2}$.

$$S_7 = \frac{S_2}{2}.S_5 + \frac{S_3}{3}.S_4 = \frac{S_2}{2}.S_5 + \frac{S_3}{3}\left(\frac{S_2}{7}\right)^2 = 5\left(\frac{S_2}{2}\right)^2.\frac{S_3}{3} + 2.\frac{S_3}{3}\left(\frac{S_2}{2}\right)$$

$$\text{or } \frac{S_7}{7} = \frac{S_5}{5}.\frac{S_2}{2}.$$

It also follows that

$$\frac{S_9}{9} = \frac{S_2}{2}.\frac{S_7}{7} + \tfrac{1}{3}\left(\frac{5_3}{3}\right)^3 \text{or} \left(\frac{S_2}{2}\right)^3\frac{S_3}{3} + \tfrac{1}{3}\left(\frac{S_3}{3}\right)^3.$$

The form of the expression for $\frac{S_n}{n}$ (putting α_2, α_3 for $\frac{S_2}{2}.\frac{S_3}{3}$) is

$$A\alpha_3{}^p\alpha_2{}^q + B\alpha_3{}^{p-2}\alpha_2{}^{q+3} + C\alpha_3{}^{p-4}\alpha_2{}^{q+6} + \ldots.$$

continued as far as the indices remain positive.

If n is of the form $6m$, $p=2m$, $q=0$.
$6m-1$, $p=2m-1$, $q=1$.
$6m+1$, $p=2m-1$, $q=2$.
$6m-2$, $p=2m-2$, $q=2$.
$6m+2$, $p=2m$, $q=1$.
$6m-3$, $p=2m-1$, $q=0$.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

(1). Put $a+b+c=0=p$, $ab+ac+bc=q$, $abc=r$.

Then $(1+ax)(1+bx)(1+cx)=1+px+qx^2+rx^3=1+qx^2+rx^3$.

Taking logarithms and equating the coefficients of x^n, we have $\frac{(-1)^{n-1}}{n}(a^n+b^n+c^n)$ for the coefficient equal to the coefficient of x^n in

$$(qx^2+rx^3) - \tfrac{1}{2}(qx^2+rx^3) + \ldots. \pm(1/n)(qx^2+rx^3)^n \ldots.$$

Let $n=2m+1$ and 2, respectively, and we get,

$$\frac{1}{2m+1}(a^{2m+1}+b^{2m+1}+c^{2m+1}) = \pm q^{m-1}r,$$

$$\frac{1}{2m-1}(a^{2m-1}+b^{2m-1}+c^{2m-1})=\mp q^{m-2}r,$$

$$\tfrac{1}{2}(a^2+b^2+c^2)=-q.$$

$$\therefore \frac{a^{2m+1}+b^{2m+1}+c^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+c^2}{2}.$$

When $m=2$, 3, we get the results in the problem.

(2). Similarly, $(1+ax)(1+bx)(1+cx)(1+dx)=1+qx^2+rx^3+sx^4$.

$\therefore \dfrac{(-1)^{n-1}}{n}(a^n+b^n+c^n+d^n)$ is equal to the coefficient of x^n in

$$(qx^2+rx^3+sx^4)-\tfrac{1}{2}(qx^2+rx^3+sx^4)^2+\ldots\ldots\pm(1/n)(qx^2+rx^3+sx^4)^n.$$

$\therefore$ As before

$$\frac{a^{2m+1}+b^{2m+1}+c^{2m+1}+d^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+c^{2m-1}+d^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+c^2+d^2}{2}.$$

The same reasoning will lead to the following :

$$\frac{a^{2m+1}+b^{2m+1}+\ldots\ldots+k^{2m+1}}{2m+1}=\frac{a^{2m-1}+b^{2m-1}+\ldots\ +k^{2m-1}}{2m-1}\cdot\frac{a^2+b^2+\ldots\ldots+k^2}{2}.$$

Also solved by *J. M. BOORMAN*, *J. SCHEFFER*, and the *PROPOSER*.

GEOMETRY.

139. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If $x^2+y^2=1$ [x and y being points corresponding to complex numbers], prove that x and y are at the ends of conjugate radii of an ellipse whose foci are ± 1. [From *Harkness and Morley's Introduction to the Theory of Functions.*]

Solution by J. W. YOUNG, Oliver Fellow in Mathematics, Cornell University, Ithica, N. Y., and FRANK GIFFIN, Assistant in Mathematics, University of Colorado, Boulder, Col.

Let $x=h+ik$, $y=m+in$.

The condition $x^2+y^2=1$, gives on equating real and imaginary parts,

$$h^2+m^2-k^2-n^2=1\ldots.(1),\ hk+mn=0\ldots.(2).$$

Now if the points (h, k) and (m, n) are the extremities of conjugate radii of an ellipse, we may write

$$\begin{matrix} h=a\cos\phi \\ k=b\sin\phi \end{matrix}\ \left\{\begin{matrix} m=a\sin\phi \\ n=-b\cos\phi \end{matrix}\right\}\ (a,\ b \text{ semi-axes of ellipse}).$$